

In honor of Roger Howe as a septuagenarian, and
in memory of Paul Sally Jr. and Joseph Shalika

COMPUTATIONS WITH BERNSTEIN PROJECTORS OF $SL(2)$

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ABSTRACT. For the p -adic group $G = SL(2)$, we present results of the computations of the sums of the Bernstein projectors of a given depth. Motivation for the computations is based on a conversation with Roger Howe in August 2013. The computations are elementary, but they provide an expansion of the delta distribution δ_{1_G} into an infinite sum of G -invariant locally integrable essentially compact distributions supported on the set of topologically unipotent elements. When these distributions are transferred, by the exponential map, to the Lie algebra, they give G -invariant distributions supported on the set of topologically nilpotent elements, whose Fourier Transforms turn out to be characteristic functions of very natural G -domains. The computations in particular rely on the $SL(2)$ discrete series character tables computed by Sally-Shalika in 1968. This new phenomenon for general rank has also been independently noticed in recent work of Bezrukavnikov, Kazhdan, and Varshavsky.

1. INTRODUCTION

A key tool in harmonic analysis on a Lie group G is the exponential map $\exp : \mathfrak{g} \rightarrow G$ from the Lie algebra \mathfrak{g} to the group G . The map, defined for all $X \in \mathfrak{g}$ is a local diffeomorphism of $0 \in \mathfrak{g}$ to $1 \in G$, and it is used to move functions and distributions, between \mathfrak{g} and G , e.g., ones which are G -invariant and eigendistributions for the center of $\mathcal{U}(\mathfrak{g})$. When F is a p -adic field, i.e., a local field of characteristic zero, and G the F -rational points of a connected reductive group defined over F , and \mathfrak{g} the Lie algebra of G , the exponential map is only defined on a certain G -invariant, open and closed subset containing $0 \in \mathfrak{g}$. In terms of Moy–Prasad filtrations [MPa, MPb], for $r \in \mathbb{R}$, set

$$\mathfrak{g}_r := \bigcup_{x \in \mathbb{B}} \mathfrak{g}_{x,r} \quad \text{and} \quad \mathfrak{g}_{r+} := \bigcup_{x \in \mathbb{B}, s > r} \mathfrak{g}_{x,s}.$$

We recall these sets are G -domains, i.e., G -invariant open and closed subsets. The set \mathfrak{g}_{0+} is the set of topologically nilpotent elements in \mathfrak{g} . The assumption $\text{char}(F) = 0$ means, there exists $R \geq 0$ so that for $r > R$, the exponential map \exp is defined on the G -domain \mathfrak{g}_r and for any $x \in \mathbb{B}$, \exp takes $\mathfrak{g}_{x,r}$ bijectively to the Moy–Prasad group $G_{x,r}$, and \mathfrak{g}_r bijectively to G_r . In the best situation R is 0.

We recall the two realizations [B] of the Bernstein center $\mathcal{Z}(G)$:

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- The geometrical realization. $\mathcal{Z}(G)$ is the algebra of G -invariant essentially compact distributions on G – a distribution is essentially compact if $\forall f \in C_0^\infty(G)$, the convolutions $D \star f$ and $f \star D$ is in $C_0^\infty(G)$.
- The spectral realization. $\mathcal{Z}(G)$ is the product $\prod_{\Omega} \mathbb{C}(\Omega)$ over the Bernstein components Ω of the algebras of (complex) regular functions $\mathbb{C}(\Omega)$.

For a fixed Bernstein component Ω , and a regular function $s \in \mathbb{C}(\Omega)$, it is known the distribution s is representable by a locally integrable function supported, and locally constant on the regular set. Of particular interest is the idempotent distribution e_Ω whose spectral realization is the constant function 1 on Ω . In the situation when G is semisimple and the Bernstein Ω corresponds to an equivalence class of supercuspidal representations, then e_Ω is the distributional character of the class times its formal degree. In this setting (G semisimple), an important result of Dat [D] states the distribution e_Ω is supported on the G -domain \mathcal{C} of compact elements, i.e., elements which belong to a compact subgroup of G . We note the set of topologically unipotent elements \mathcal{U}^{top} is contained in \mathcal{C} .

Based on known examples, e.g., for $\text{SL}(2)$ see [MT1, MTb], if $s \in \mathbb{C}(\Omega)$ is viewed as a distribution, its support is generally not contained in \mathcal{U}^{top} . An intriguing question is whether it is possible to find an element in the span of finitely many $\mathbb{C}(\Omega)$ whose support is contained in \mathcal{U}^{top} .

In [MPa, MPb], the depth invariant $\rho(\pi)$ is defined for any irreducible smooth representation π , and it is known the depth is the same for all irreducible representation (classes) occurring in a Bernstein component. Thus, a depth $\rho(\Omega)$ is attached to any Bernstein component Ω .

Let $d \geq 0$ be the depth of a Bernstein component, and set

$$e_d := \sum_{\rho(\Omega)=d} e_\Omega \quad \text{and} \quad \sigma_d := \sum_{\rho(\Omega) \leq d} e_\Omega .$$

Here we show, for $G = \text{SL}(2, F)$, the Bernstein center element σ_d has support in the topological unipotent set

$$\mathcal{U}_{d^+}^{\text{top}} = \bigcup_{x \in \mathbb{B}} G_{x, d^+}$$

Recall (i) an element $y \in G$ is called split (resp. elliptic), if its characteristic polynomial has distinct roots in F (resp. not in F), and (ii) the depth of an irreducible representations is a half-integer, i.e., in $\frac{1}{2}\mathbb{N}$.

Theorem. 8.2. *Suppose F is a p -adic field with odd residue characteristic, and $G = \text{SL}(2, F)$. For $d \in \frac{1}{2}\mathbb{N}$, set $d^+ := k + \frac{1}{2}$. Then, we have $\text{supp}(\sigma_d) \subset \mathcal{U}_{d^+}^{\text{top}}$, and on $\mathcal{U}_{d^+}^{\text{top}}$:*

- *When d is integral:*

$$\sigma_d(y) = (q^2 - 1)q^{3d} \begin{cases} \left(\frac{2q^{-d}}{|\alpha - \alpha^{-1}|_F} - 1 \right) & \text{when } y \text{ is split with eigenvalues } \alpha, \alpha^{-1} \\ -1 & \text{when } y \text{ is elliptic} \end{cases}$$

- When d is half-integral:

$$\sigma_d(y) = (q^2 - 1) q^{3d + \frac{1}{2}} \begin{cases} \left(\frac{2q^{-(d+\frac{1}{2})}}{|\alpha - \alpha^{-1}|_F} - 1 \right) & \text{when } y \text{ is split with eigenvalues } \alpha, \alpha^{-1} \\ -1 & \text{when } y \text{ is elliptic} \end{cases}$$

We observe, for $\mathrm{SL}(2)$, the projector σ_0 is equal to the Steinberg character restricted to the topological unipotent set.

As mentioned above, there is an $R \geq 0$ so that the exponential power series $\exp(X)$ is convergent when the eigenvalues of X have normalized valuations greater than R , and for $r > R$, the exponential map is then a bijection between the set $\mathcal{N}_r^{\mathrm{top}} \subset \mathfrak{g}$, and the set $\mathcal{U}_r^{\mathrm{top}} \subset G$. In this ideal situation, we can then move the distributions in (8.2) to the Lie algebra. For $d \in \frac{1}{2}\mathbb{N}$, satisfying $d > R$, the Lie algebra distribution $\sigma_d \circ \exp$ has support in \mathfrak{g}_{d+} , and we have the homogeneity relation

$$(\sigma_{d+1} \circ \exp)(\varpi Y) = q^3 (\sigma_d \circ \exp)(Y). \quad (1.1)$$

Whence, their Fourier transforms satisfy the homogeneity relation

$$\mathrm{FT}(\sigma_{k+1} \circ \exp)(\varpi^{-1}Y) = \mathrm{FT}(\sigma_k \circ \exp)(Y). \quad (1.2)$$

In this regard, we show in the appendix the following:

Proposition. A.1. *For $\mathfrak{sl}(2)$, we have*

- The Fourier transforms $\mathrm{FT}(1_{\mathfrak{g}_0})$ and $\mathrm{FT}(1_{\mathfrak{g}_{-\frac{1}{2}}})$ have support in the sets $\mathfrak{g}_{0+} := \mathfrak{g}_{\frac{1}{2}}$ and $\mathfrak{g}_{(\frac{1}{2})+} := \mathfrak{g}_1$ respectively.
In particular, the support is contained in $\mathcal{N}^{\mathrm{top}}$.
- For $k \geq 1$, the Fourier transform $\mathrm{FT}(1_{\mathfrak{g}_{-k}})$ has support in $\mathfrak{g}_{k+} := \mathfrak{g}_{k+\frac{1}{2}}$.

For a general connected reductive p-adic group, when $R = 0$ and other conditions, Kim [Ka,Kb], showed, for X in $\mathfrak{g}_{(\frac{d}{2})+}$:

$$\int_{\widehat{G}_{\leq d}^{\mathrm{temp}}} \Theta_{\pi}(\exp(X)) d\mu_{\mathrm{PM}}(\pi) = \mathrm{FT}(1_{\mathfrak{g}_{-d}})(X).$$

The integral is over the (classes of) irreducible tempered representations of depth less than or equal to d . Thus, for $\mathrm{SL}(2)$, when $R = 0$ we have

$$\sigma_d \circ \exp = \mathrm{FT}(1_{\mathfrak{g}_{-d}}) \quad (\text{both sides have support in } \mathfrak{g}_{d+}).$$

We conjecture, for $\mathrm{SL}(2)$, and more generally for any connected reductive p-adic group, this identity to be true even when $R > 0$, as long as $d > R$.

In October 2014, through correspondence with Roman Bezrukavnikov, the author became aware of unpublished work in-progress of Bezrukavnikov, Kazhdan, and Varshavsky in which

they independently discovered and proved, for a general connected reductive p -adic group, the support of the projector σ_d is in the topological unipotent set, that σ_0 is the restriction of the Steinberg character to the unipotent set, and the identification of the Fourier transforms $\text{FT}(\sigma_d \circ \exp)$. A preprint [BKV] of their work became available in April 2015.

Motivation for considering the support of the depth zero projector e_0 , the sum of the projectors e_Ω with $\rho(\Omega) = 0$, aroused during a conversation the author had with Roger Howe in August 2013, and the author successfully verified the support is contained in $\mathcal{U}_{0+}^{\text{top}}$ and a formula for the values in December 2013. Extension of the support and values of σ_d to all depths was completed in March 2015 while the author was a visiting faculty at the University of Utah. The author kindly thanks the hospitality of the Mathematics Department of the University of Utah, with special thanks to Dragan Milićić. The author gratefully acknowledges useful conversations with Roman Bezrukavnikov, Roger Howe, Ju-Lee Kim, and Fiona Murnaghan. The author gave workshop talks of the case e_0 at the Mathematical Research Institute of Oberwolfach, the University of Zagreb, and the University of Chicago, and thanks these institutions for their invitations.

2. NOTATION

We set some notation. Let F denote a p -adic field (so of characteristic zero). Let \mathcal{R}_F denote the ring of integers of F , let \wp_F its prime ideal, and let ϖ be a prime element. Set $\mathbb{F}_q = \mathcal{R}_F/\wp_F$ to be the residue field. To be able to use the Sally-Shalika character tables [SS], we assume the residue characteristic of F is odd.

Let \mathcal{B} be the Bruhat-Tits building $G = \text{SL}(2, F)$. The group $\text{GL}(2, F)$, whence also G , acts on \mathcal{B} . There are respectively two G -orbits, and one $\text{GL}(2, F)$ -orbit of vertices in \mathcal{B} . The maximal compact subgroups of G are precisely the stabilizer subgroups G_x (in G) of vertices x in \mathcal{B} ; whence, there two conjugacy classes of maximal compact subgroups in G . Let x_0 and x_1 be the vertices in \mathcal{B} so that

$$G_{x_0} = \text{SL}(2, \mathcal{R}_F) \quad , \quad \text{and} \quad G_{x_1} = \begin{bmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{bmatrix} \text{SL}(2, \mathcal{R}_F) \begin{bmatrix} \varpi & 0 \\ 0 & 1 \end{bmatrix} \quad (2.1)$$

are the familiar representatives of the two conjugacy classes of maximal compact subgroups of G . If e is an edge (with vertices y and z and (midpoint) barycenter $b(e)$) in \mathcal{B} , then the stabilizer (in G) of e is an Iwahori subgroup equal to $G_y \cap G_z = G_{b(e)}$.

We note the two vertices x_0 and x_1 mentioned in (2.1) are the vertices of an edge $e_{01} \in \mathcal{B}$. Let $x_{01} = b(e_{01})$ – the barycenter of e_{01} . The Iwahori subgroup $G_{x_{01}}$ equals:

$$G_{x_{01}} = G_{x_0} \cap G_{x_1} = \{ g \in \text{SL}(2, \mathcal{R}_F) \mid g \text{ upper triangular mod } \wp_F \} . \quad (2.2)$$

For notational convenience, we set

$$\mathcal{K} = G_{x_0} \quad , \quad \mathcal{K}' = G_{x_1} \quad , \quad \mathcal{I} = \mathcal{K} \cap \mathcal{K}' = G_{x_{01}} \quad .$$

Set $\mathfrak{g} = \mathfrak{sl}(2, F)$. For any $x \in \mathcal{B}$, let

$$G_{x,r} \quad \text{and} \quad r \geq 0 \quad \text{and} \quad \mathfrak{g}_{x,r} \quad \text{and} \quad r \in \mathbb{R}$$

be the Moy–Prasad filtration subgroups. We recall:

- (i) If $x \in \mathcal{B}$ is a vertex, then the jumps in the filtration subgroups $G_{x,r}$ occur at integral r , i.e.,

$$G_{x,r^+} = G_{x,r+1} \quad \text{when} \quad r \in \mathbb{N}$$

- (ii) If $x = b(e)$ is the barycenter of an edge e , then the jumps in the filtration subgroups occur at half-integral r , i.e.,

$$G_{x,r^+} = G_{x,r+\frac{1}{2}} \quad \text{when} \quad r \in \frac{1}{2}\mathbb{N} \quad ,$$

and similarly for $\mathfrak{g}_{x,r}$. For the latter, we have

$$\mathfrak{g}_{x,r+1} = \varpi \mathfrak{g}_{x,r} \quad .$$

Take ψ to be an additive character of F which has conductor \wp_F .

- (iii) For $2r \geq s$, the quotient group $G_{x,r}/G_{x,s}$ is abelian and canonically isomorphic to $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}$. The residual characteristic is odd assumption means the trace pairing

$$\mathfrak{g}_{x,r}/\mathfrak{g}_{x,s} \times \mathfrak{g}_{x,-s^+}/\mathfrak{g}_{x,-r^+} \rightarrow F/\wp_F$$

allows us to identify the Pontryagin dual $(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,s})^\wedge$ (whence $(G_{x,r}/G_{x,s})^\wedge$) with $\mathfrak{g}_{x,-s^+}/\mathfrak{g}_{x,-r^+}$. A coset $\Xi = X + \mathfrak{g}_{x,-r^+}$ yields the character ψ_Ξ of $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}$ given as:

$$\psi_\Xi(Y) := \psi(\mathrm{trace}(XY)) \quad .$$

We note:

- For the vertex x_0 :

$$\mathfrak{g}_{x_0,0}/\mathfrak{g}_{x_0,1} = \mathfrak{sl}(2, \mathbb{F}_q) \quad .$$

For a general vertex $x \in \mathcal{B}$ a vertex, and $r \in \mathbb{N}$:

$$(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,(r+1)})^\wedge = \mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r+1)} \simeq \mathfrak{g}_{x,0}/\mathfrak{g}_{x,1} \simeq \mathfrak{sl}(2, \mathbb{F}_q)$$

The isomorphism $\mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r+1)} \simeq \mathfrak{g}_{x,0}/\mathfrak{g}_{x,1}$ is the natural one given by scalar multiplication by ϖ . The isomorphism $\mathfrak{g}_{x,0}/\mathfrak{g}_{x,1} \simeq \mathfrak{sl}(2, \mathbb{F}_q)$ is up to a conjugation by an element of $\mathrm{GL}(2, \mathbb{F}_q)$. A coset $X \in \mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r+1)}$ is, by definition, non-degenerate if, as an element in $\mathfrak{sl}(2, \mathbb{F}_q)$, it is non-nilpotent.

- For the barycenter point x_{01} :

$$\mathfrak{g}_{x_{01}, \frac{1}{2}} / \mathfrak{g}_{x_{01}, 1} = \left\{ \begin{bmatrix} \varpi a & b \\ \varpi c & -\varpi a \end{bmatrix} \mid a, b, c \in \mathcal{R}_F \right\} / \left\{ \begin{bmatrix} \varpi a & \varpi b \\ \varpi^2 c & -\varpi a \end{bmatrix} \mid a, b, c \in \mathcal{R}_F \right\}.$$

For $d \in \mathbb{N}$:

$$(\mathfrak{g}_{x_{01}, d+\frac{1}{2}} / \mathfrak{g}_{x_{01}, (d+1)})^\wedge = \mathfrak{g}_{x_{01}, -d-\frac{1}{2}} / \mathfrak{g}_{x_{01}, -d}$$

We recall a coset

$$X = \varpi^{-(d+1)} \begin{bmatrix} \varpi a & b \\ \varpi c & -\varpi a \end{bmatrix} + \mathfrak{g}_{x_{01}, -d} \in \mathfrak{g}_{x_{01}, -d-\frac{1}{2}} / \mathfrak{g}_{x_{01}, -d} \quad (a, b, c \in \mathcal{R}_F), \quad (2.3)$$

is non-degenerate if b and c are both units.

3. REVIEW OF EARLIER RESULTS

3.1. A result of J. Dat on support. Suppose G is a general connected reductive p -adic group. A *compact element* $\gamma \in G$ is one which lies in a compact subgroup of G . Set

$$\mathcal{C} := \text{set of compact elements.}$$

The following important result of Dat says the support of a projector e_Ω to a Bernstein component Ω is contained in \mathcal{C} .

Theorem 3.1.1. (Dat, [D]) *Suppose G is a connected reductive p -adic group. Let \mathcal{C} be the set of compact elements in G . For any Bernstein component Ω , let e_Ω be the element of the Bernstein center which projects onto the component Ω . Then*

$$\text{supp}(e_\Omega) \subset \mathcal{C}.$$

3.2. A result of Moy–Prasad on depths. If π is an irreducible smooth representation of a connected reductive p -adic group, let $\rho(\pi)$ denote its depth as defined in [MPa, MPb]. We recall the following result on depths of representations which implies we can define the depth of a Bernstein component.

Theorem 3.2.1. (Moy–Prasad, [MPb], Theorem 5.2) *Suppose G is a connected reductive p -adic group, MN is a parabolic subgroup, and σ an irreducible smooth representation of M . If π is any irreducible subquotient of $\text{Ind}_{MN}^G(\sigma)$, then $\rho(\pi) = \rho(\sigma)$.*

Let $d \geq 0$ be the depth of a Bernstein component, and set

$$e_d := \sum_{\rho(\Omega)=d} e_\Omega \quad \text{and} \quad \sigma_d := \sum_{\rho(\Omega) \leq d} e_\Omega.$$

As mentioned in the introduction, when $G = \text{SL}(2)$, in the remainder of this manuscript, we show σ_d has support in the topological unipotent set \mathcal{U}^{top} , and indeed in the smaller G -domain $\mathcal{U}_{d^+}^{\text{top}}$.

3.3. A partition of the compact elements \mathcal{C} of $\mathrm{SL}(2)$. We partition \mathcal{C} into three subsets:

$$\mathcal{C} = \mathcal{U}^{\mathrm{top}} \coprod -\mathrm{I}_{2 \times 2} \mathcal{U}^{\mathrm{top}} \coprod \mathcal{C}_{\mathrm{st-reg}}$$

Here, $\mathcal{C}_{\mathrm{st-reg}}$ is the set of ‘strongly regular’ elements, i.e., those elements whose eigenvalues are not congruent to each other modulo the prime.

4. PRINCIPAL SERIES PROJECTORS FOR $\mathrm{SL}(2)$

For $\mathrm{SL}(2)$, Moy–Tadić [MT1, MTb] explicitly computed the projectors e_Ω for principal series components. To state the results, we normalize Haar measure on $\mathrm{SL}(2)$ so that

$$\mathrm{meas}(\mathrm{SL}(2, \mathcal{R}_F)) = 1. \quad (4.1)$$

We enumerate the Principal Series Bernstein components as: (i) $\Omega(\{\chi, \chi^{-1}\})$, where χ is a character of \mathcal{R}_F^\times with $\chi \neq \chi^{-1}$, (ii) $\Omega(\mathrm{sgn})$, with sgn the order two character of \mathcal{R}_F^\times , and (iii) $\Omega = \Omega_{\mathrm{triv}}$, the Bernstein component of irreducible representations with non-zero Iwahori-fixed vectors. Let $f(\chi)$ denote the conductor of χ . The Principal Series projectors are:

Regular PS $f(\chi) = d + 1$

$$e_{\Omega(\{\chi, \chi^{-1}\})}(y) = \begin{cases} (q+1) q^d \frac{\chi(\alpha) + \chi(\alpha^{-1})}{|\alpha - \alpha^{-1}|_F}, & y \text{ split with eigenvalues } \alpha, \alpha^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Sgn PS

$$e_{\Omega(\mathrm{sgn})}(y) = \begin{cases} (q+1) \frac{\mathrm{sgn}(\alpha)}{|\alpha - \alpha^{-1}|_F}, & y \text{ split with eigenvalues } \alpha, \alpha^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Unramified PS (Iwahori fixed vectors)

$$e_\Omega(y) = \begin{cases} \frac{2q}{|\alpha - \alpha^{-1}|_F} - (q-1), & y \text{ split with eigenvalues } \alpha, \alpha^{-1} \\ -(q-1) & y \text{ elliptic} \end{cases}$$

Let e_d^{PS} be the sum of the principal series depth d Bernstein projectors.

For $d = 0$ we have:

$$e_0^{\text{PS}}(y) = (q-1) \begin{cases} \frac{(q+2)}{|\alpha-\alpha^{-1}|} - 1 & \text{when } y \in \mathcal{U}^{\text{top}} \text{ is split with eigenvalues } \alpha, \alpha^{-1} \\ \frac{1}{|\alpha-\alpha^{-1}|} - 1 & \text{when } y \in -\text{I}_{2 \times 2} \mathcal{U}^{\text{top}} \text{ is split with eigenvalues } \alpha, \alpha^{-1} \\ 0 & \text{when } y \in \mathcal{C}_{\text{st-reg}} \text{ is split} \\ -1 & \text{when } y \in \mathcal{C} \text{ is elliptic} \end{cases} \quad (4.2)$$

For $d > 0$, each principal series Bernstein projector e_Ω has support in the split set and consequently the support of e_d^{PS} is in the split set too. For $y \in \text{SL}(2)$ a (regular) split element, let α, α^{-1} be the eigenvalues of y . We have:

$$e_d^{\text{PS}}(y) = (q+1)q^d(q-1)q^{d-1} \begin{cases} \frac{(-1)}{|\alpha-\alpha^{-1}|} = -q^d & \text{when } \alpha, \alpha^{-1} \text{ in } (1 + \wp_F^d) \setminus (1 + \wp_F^{d+1}) \\ \frac{(q-1)}{|\alpha-\alpha^{-1}|} & \text{when } \alpha, \alpha^{-1} \text{ in } (1 + \wp_F^{d+1}) \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

5. THE PROJECTOR e_0

Set

$$e_0^{\text{cusp}} = \sum_{\substack{\pi \text{ cuspidal} \\ \text{depth } 0}} e_\pi. \quad (5.1)$$

In this section, we determine, with the aid of the Sally-Shalika tables [SS], the values of e_0^{cusp} . To conveniently use their tables we use their normalization of Haar measure, i.e., $\text{meas}(G_{x_0}) = 1$, as in (4.1).

5.1. Irreducible cuspidal representations of depth zero. Let $\mathcal{K} = G_{x_0}$, and $\mathcal{K}' = G_{x_1}$. We recall $\mathcal{K}/G_{x_0,1} \simeq \text{SL}(2, \mathbb{F}_q) \simeq \mathcal{K}'/G_{x_1,1}$.

Proposition 5.1.1.

- (i) *If κ is an irreducible cuspidal representation of $\text{SL}(2, \mathbb{F}_q)$, let κ_{x_0} and κ_{x_1} denote its inflation to $\mathcal{K} = G_{x_0}$ and $\mathcal{K}' = G_{x_1}$ respectively. Then, the representations $\text{c-Ind}_{\mathcal{K}}^G(\kappa_{x_0})$ and $\text{c-Ind}_{\mathcal{K}'}^G(\kappa_{x_1})$ are irreducible supercuspidal representations of G . Furthermore, the supercuspidal representations induced from the group \mathcal{K} are inequivalent to those induced from the group \mathcal{K}' .*

- (ii) Any irreducible supercuspidal representation (π, V_π) of depth zero is equivalent to a $\mathrm{c}\text{-Ind}_{\mathcal{K}}^G(\kappa_{x_0})$ or a $\mathrm{c}\text{-Ind}_{\mathcal{K}'}^G(\kappa_{x_1})$
- (iii) Normalize Haar measure on $\mathrm{SL}(2)$ so $\mathrm{meas}(\mathcal{K}) = 1$. If κ is a cuspidal representation of $\mathrm{SL}(2, \mathbb{F}_q)$ inflated to \mathcal{K} , and $\pi = \mathrm{c}\text{-Ind}_{\mathcal{K}}^G(\kappa)$, then

$$\text{the formal degree } d_\pi = \text{degree}(\kappa) .$$

Similarly for \mathcal{K}' .

We briefly review the cuspidal representations of $\mathrm{SL}(2, \mathbb{F}_q)$:

- Let $T \subset \mathrm{SL}(2, \mathbb{F}_q)$ be the (elliptic) torus of order $(q+1)$. Take β to be a primitive root of T , set $\zeta = e^{\frac{2\pi\sqrt{-1}}{(q+1)}}$, and for integral $0 \leq i \leq q$, let $\theta_i \in \widehat{T}$ be the character $\theta_i(\beta) = \zeta^i$. For $0 < i$, the character θ_i is conjugate under the normalizer $N_G(T)$ to $\theta_{(q+1)-i}$.
- For $i = 1, 2, \dots, q$, there is a cuspidal representation σ_i of $\mathrm{SL}(2, \mathbb{F}_q)$ of dimension $(q-1)$. Let χ_i be the character of σ_i . Then, $\chi_i = \chi_{(q+1)-i}$. For $i \neq \frac{(q+1)}{2}$, the character is irreducible, and for $i = \frac{(q+1)}{2}$, the character is the sum of two irreducible characters η_1, η_2 of degree $\frac{q-1}{2}$. Let $\epsilon \in \mathbb{F}_q^\times$ be a non-square. In all cases, the support of χ_i is on the elements:

$$\mathrm{Id} , \quad -\mathrm{Id} , \quad \beta^k , \quad u_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} , \quad u_\epsilon = \begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix} , \quad -u_1 , \quad -u_\epsilon \quad (5.1.2)$$

with values:

	Id	-Id	β^k	u_1	u_ϵ	$-u_1$	$-u_\epsilon$
χ_i	$(q-1)$	$(-1)^i(q-1)$	$-(\zeta^{ik} + \zeta^{-ik})$	-1	-1	$(-1)^{(i+1)}$	$(-1)^{(i+1)}$

Set $\pi_i := \mathrm{c}\text{-Ind}_{\mathcal{K}}^G(\sigma_i)$, and let Θ_i be the character of π_i . Note that $\pi_i(-\mathrm{Id})$ is $(-1)^i$ times the identity operator. If we use \mathcal{K}' instead of \mathcal{K} , we can define analogous representations π'_i and characters Θ'_i . Set

$$\begin{aligned} e_{\mathcal{K}} &:= (q-1) \sum_{1 \leq i \leq \frac{(q-1)}{2}} \Theta_i + \frac{(q-1)}{2} \Theta_{(\frac{q+1}{2})} = \frac{(q-1)}{2} \sum_{1 \leq i \leq q} \Theta_i \\ e_{\mathcal{K}'} &:= (q-1) \sum_{1 \leq i \leq \frac{(q-1)}{2}} \Theta'_i + \frac{(q-1)}{2} \Theta'_{(\frac{q+1}{2})} = \frac{(q-1)}{2} \sum_{1 \leq i \leq q} \Theta'_i , \end{aligned} \quad (5.1.3)$$

so $e_0 = e_0^{\mathrm{PS}} + e_{\mathcal{K}} + e_{\mathcal{K}'}$.

5.2. e_0 on split regular elements. Suppose $y \in G$ is a regular split compact element with eigenvalues α, α^{-1} . From Tables 2 and 3 (pages 1235-1236) of Sally-Shalika [SS]:

$$\Theta_i(y) = \begin{cases} 0 & \text{when } \alpha \not\equiv \alpha^{-1} \pmod{\wp} \\ \frac{1}{|\alpha - \alpha^{-1}|} - 1 & \text{when } \alpha \equiv 1 \pmod{\wp} \end{cases} \quad (5.2.1)$$

Similarly for the character Θ'_i . We deduce

$$e_K(y) = \frac{(q-1)}{2} \begin{cases} 0 & \text{when } \alpha \not\equiv \alpha^{-1} \pmod{\wp} \\ (-1) \left(\frac{1}{|\alpha - \alpha^{-1}|} - 1 \right) & \text{when } \alpha \equiv -1 \pmod{\wp} \\ q \left(\frac{1}{|\alpha - \alpha^{-1}|} - 1 \right) & \text{when } \alpha \equiv 1 \pmod{\wp} , \end{cases} \quad (5.2.2)$$

and whence

$$e_0(y) = (q-1)(q+1) \begin{cases} 0 & \text{unless } \alpha \equiv 1 \pmod{\wp} \\ \left(\frac{2}{|\alpha - \alpha^{-1}|} - 1 \right) & \text{when } \alpha \equiv 1 \pmod{\wp} . \end{cases} \quad (5.2.3)$$

5.3. e_0 on ramified elliptic elements. Suppose $y \in G$ is a ramified elliptic element, i.e., its eigenvalues α, α^{-1} belong to a ramified quadratic extension E . We note either y or $-y$ is topologically unipotent. We assume y topologically unipotent, i.e., $\alpha \equiv 1 \pmod{\wp_E}$. From Table 2 (page 1235) of Sally-Shalika [SS]:

$$\Theta_i(y) = -1 , \quad \text{therefore} \quad \Theta_i(-y) = -1(-1)^i , \quad (5.3.1)$$

and so

$$e_0^{\text{cusp}}(y) = -(q-1)q \quad \text{and} \quad e_0^{\text{cusp}}(-y) = (q-1) . \quad (5.3.2)$$

We deduce

$$\frac{1}{(q-1)(q+1)} e_0(y) = \begin{cases} 0 & \text{when } y \text{ is not topologically unipotent} \\ -1 & \text{when } y \text{ is topologically unipotent} . \end{cases} \quad (5.3.3)$$

5.4. e_0 on unramified elliptic elements. Suppose $y \in G$ is a unramified elliptic element, i.e., its eigenvalues α, α^{-1} belong to a unramified quadratic extension E . We consider the two cases depending on whether the two eigenvalues are congruent modulo \wp_E .

Case $\alpha \not\equiv \alpha^{-1} \pmod{\wp_E}$: We note there are two G -conjugacy classes of elements which have eigenvalues α, α^{-1} . One class has non-empty intersection with \mathcal{K} , and empty intersection with \mathcal{K}' , and vice versa for the other class. We assume $y \in \mathcal{K}$, and find, using Harish-Chandra's formula [HC] for the character of the compactly induced representation:

$$e_{\mathcal{K}}(y) = \frac{(q-1)}{2} \text{c-Ind}_{\mathcal{K}}^G \left(\sum_{1 \leq i \leq q} \Theta_{\sigma_i} \right) (y) = (q-1) \ , \text{ and } e_{\mathcal{K}'}(y) = 0 \ .$$

If $y \in \mathcal{K}'$, we have the obvious analogous situation. Therefore,

$$e_0(y) = (e_0^{\text{PS}} + e_{\mathcal{K}} + e_{\mathcal{K}'})(y) = (q-1) - (q-1) = 0 \quad \text{when } \alpha \not\equiv \alpha^{-1} \pmod{\wp_E} \ .$$

Case $\alpha \equiv \alpha^{-1} \pmod{\wp_E}$: Here $\alpha \equiv \pm 1 \pmod{\wp_E}$. We assume $\alpha \equiv 1$, so y is topologically unipotent. Let sgn_E denote the class-field character of F^\times of the unramified quadratic extension E . By Table 2 of Sally-Shalika [SS], the values of $\Theta_i(y)$ and $\Theta'_i(y)$ in some order are:

$$-1 \pm \text{sgn}_E(\gamma) \frac{1}{|\alpha - \alpha^{-1}|} \ .$$

Here $\gamma = \pm\alpha - \alpha^{-1}$ is the (imaginary) part of α . Note $\text{sgn}_E(\pm 1) = 1$. So,

$$(\Theta_i + \Theta'_i)(y) = 2 \quad \text{and} \quad (\Theta_i + \Theta'_i)(-y) = (-1)^i 2 \ .$$

It follows:

$$\begin{aligned} (e_{\mathcal{K}} + e_{\mathcal{K}'})(y) &= \frac{(q-1)}{2} q(-2) = -(q-1)q \ , \text{ and} \\ (e_{\mathcal{K}} + e_{\mathcal{K}'})(-y) &= \frac{(q-1)}{2} (-1)(-2) = (q-1) \ . \end{aligned}$$

Thus,

$$e_0(y) = -(q-1)(q+1) \ , \text{ and } e_0(-y) = 0 \quad \text{when } \alpha \equiv 1 \pmod{\wp_E} \ .$$

5.5. Summary Table of e_0 .

Table 1 depth zero	
y has eigenvalues α, α^{-1} : value of $\frac{1}{(q-1)(q+1)} e_0(y)$	
y split	$\begin{cases} 0 & \text{when } \alpha \not\equiv 1 \pmod{\wp} \\ \left(\frac{2}{ \alpha - \alpha^{-1} } - 1 \right) & \text{when } \alpha \equiv 1 \pmod{\wp} . \end{cases}$
y elliptic or unramified	$\begin{cases} 0 & \text{when } y \text{ is not topologically unipotent} \\ -1 & \text{when } y \text{ is topologically unipotent} . \end{cases}$

6. INTEGRAL DEPTH SUPERCUSPIDAL REPRESENTATIONS

6.1. The partition of supercuspidal representations by the groups \mathcal{K} and \mathcal{K}' . We abbreviate the filtration subgroups $\mathcal{K} = G_{x_0, r}$ as \mathcal{K}_r ($r \in \mathbb{N}$), and similarly we abbreviate the filtration subgroups of $\mathcal{K}' = G_{x_1}$.

Suppose d is a positive integer. We recall if (π, V_π) is an irreducible supercuspidal representation (π, V_π) of depth d , then either

$$V_\pi^{\mathcal{K}_{d+1}} \neq \{0\} \quad , \quad (\text{exclusive}) \quad \text{or} \quad V_\pi^{\mathcal{K}'_{d+1}} \neq \{0\}$$

In what follows, we assume $V_\pi^{\mathcal{K}_{d+1}} \neq \{0\}$ and remark the obvious transposition of results holds when $V_\pi^{\mathcal{K}'_{d+1}} \neq \{0\}$.

The subgroup \mathcal{K}_{d+1} is normal in \mathcal{K} , and therefore $V_\pi^{\mathcal{K}_{d+1}}$ is \mathcal{K} -invariant, and so \mathcal{K}_d -invariant. We note $(\mathcal{K}_d/\mathcal{K}_{d+1})^\wedge = \mathfrak{g}_{x_0, -d}/\mathfrak{g}_{x_0, (-d+1)} \simeq \mathfrak{sl}(2, \mathbb{F}_q)$, and we recall:

- (i) If (π, V_π) is an irreducible supercuspidal representation of depth d with $V_\pi^{\mathcal{K}_{d+1}} \neq \{0\}$, then the characters of $\mathcal{K}_d/\mathcal{K}_{d+1}$ which appear in $V_\pi^{\mathcal{K}_{d+1}}$ have the form ϕ_Ξ where the coset $\Xi \in \mathfrak{g}_{x_0, -d}/\mathfrak{g}_{x_0, (-d+1)} \simeq \mathfrak{sl}(2, \mathbb{F}_q)$ is an elliptic element in $\mathfrak{sl}(2, \mathbb{F}_q)$, i.e., has irreducible characteristic polynomial. By Clifford theory, the set of ϕ_Ξ 's which appear in $V_\pi^{\mathcal{K}_{d+1}}$ form a single \mathcal{K} -orbit in $\mathfrak{g}_{x_0, -d}/\mathfrak{g}_{x_0, (-d+1)}$.

- (ii) The Adjoint action of \mathcal{K} on $\mathfrak{sl}(2, \mathbb{F}_q)$ has $\frac{(q-1)}{2}$ orbits of elliptic elements, and each orbit contains $(q-1)q$ elliptic elements.
- (iii) If ϕ_Ξ is a character of $\mathcal{K}_d/\mathcal{K}_{d+1}$ attached to an elliptic coset of $\Xi \in \mathfrak{g}_{x_0, -d}/\mathfrak{g}_{x_0, (-d+1)} \simeq \mathfrak{sl}(2, \mathbb{F}_q)$, then

$$\mathrm{c}\text{-Ind}_{\mathcal{K}_d}^G (\phi_\Xi)$$

is a finite length (completely reducible) supercuspidal representation. If (π, V_π) is as in part (i), i.e., $V_\pi^{\mathcal{I}_{d^+}}$ contains the (non-degenerate) character ϕ_Ξ , then by Frobenius reciprocity:

$$\mathrm{Hom}_G(V_\pi, \mathrm{c}\text{-Ind}_{\mathcal{K}_d}^G (\phi_\Xi)) \neq \{0\}.$$

Furthermore:

- Up to isomorphism, $\mathrm{c}\text{-Ind}_{\mathcal{K}_d}^G (\phi_\Xi)$ contains $(q+1)q^{(d-1)}$ distinct classes of irreducible supercuspidal representations (σ, V_σ) . In particular, the number of distinct irreducible supercuspidal representations of depth d which are induced from \mathcal{K} is:

$$\frac{(q-1)}{2} (q+1) q^{(d-1)} \quad (6.1.1)$$

- The formal degree of any irreducible (σ, V_σ) occurring in $\mathrm{c}\text{-Ind}_{\mathcal{K}_d}^G (\phi_\Xi)$ is:

$$d_\sigma = \frac{1}{\mathrm{meas}(\mathcal{K})} (q-1) q q^{(d-1)} \quad (6.1.2)$$

- The multiplicity in $\mathrm{c}\text{-Ind}_{\mathcal{K}_d}^G (\phi_\Xi)$ of any (σ, V_σ) occurring in it is $q^{(d-1)}$; in particular is independent of σ .
- (iv) For an elliptic character ϕ_Ξ of $\mathcal{K}_d/\mathcal{K}_{d+1}$, let S_Ξ denote the set of $(q+1)q^{(d-1)}$ classes of irreducible supercuspidal representations occurring in $\mathrm{c}\text{-Ind}_{\mathcal{K}_d}^G (\phi_\Xi)$, and let Θ_Ξ denote the character of the representation $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G (\phi_\Xi)$. We have:

$$\Theta_\Xi = q^{(d-1)} \sum_{\sigma \in S_\Xi} \Theta_\sigma.$$

By Harish-Chandra's character formula [HC] for induction from an open compact subgroup, Θ_Ξ is supported on $\mathrm{Ad}(G)(\mathcal{K}_d)$. If the cosets Ξ_1 and Ξ_2 belong to the same \mathcal{K} -orbit, then $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G (\phi_{\Xi_1})$ and $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G (\phi_{\Xi_2})$ are equivalent representations, and therefore $\Theta_{\Xi_1} = \Theta_{\Xi_2}$.

Set

$$\tau := \sum_{\Xi} \phi_\Xi \quad \text{the sum of the elliptic characters of } \mathcal{K}_d \text{ (modulo } \mathcal{K}_{d+1}), \quad (6.1.3)$$

and let Θ_τ denote the character of the induced representation $\mathrm{c}\text{-Ind}_{\mathcal{K}_d}^G(\tau)$. We have:

$$\Theta_\tau = \sum_{\Xi} \Theta_\Xi = \sum_{\Xi} q^{(d-1)} \sum_{\sigma \in S_\Xi} \Theta_\sigma = (q-1) q q^{(d-1)} \sum_{\substack{\mathcal{K} \\ \rho(\sigma)=d}} \Theta_\sigma, \quad (6.1.4)$$

and so,

$$\frac{\Theta_\tau}{\text{meas}(\mathcal{K})} = \frac{(q-1)q^{d-1}}{\text{meas}(\mathcal{K})} \sum_{\rho(\sigma)=d} \Theta_\sigma = \sum_{\rho(\sigma)=d} d_\sigma \Theta_\sigma . \quad (6.1.5)$$

The sum $\sum_{\mathcal{K}}$ is over the classes of irreducible supercuspidal representations of depth d which can be induced from \mathcal{K} .

For the compact group \mathcal{K}' , with the obviously transposed construction, we have the similar identity:

$$\frac{\Theta_{\tau'}}{\text{meas}(\mathcal{K}')} = \sum_{\rho(\sigma)=d} d_\sigma \Theta_\sigma . \quad (6.1.6)$$

Whence,

$$e_d^{\text{cusp}} = \sum_{\rho(\sigma)=d} d_\sigma \Theta_\sigma + \sum_{\rho(\sigma)=d} d_\sigma \Theta_\sigma = \frac{1}{\text{meas}(\mathcal{K})} (\Theta_\tau + \Theta_{\tau'}) \quad (6.1.7)$$

6.2. The projector e_d^{cusp} . Let Z denote the subgroup of scalar matrices in $\text{GL}(2, F)$, and set

$$\begin{aligned} K &:= \text{GL}(2, \mathcal{R}_F) \\ \mathfrak{k} &:= \mathfrak{gl}(2, \mathcal{R}_F) \\ G_u &:= \{ g \in \text{GL}(2, F) \mid \text{val}_F(\det(g)) \text{ is even} \} = Z K \text{SL}(2, F) . \end{aligned} \quad (6.2.1)$$

Let K_n be the usual congruence subgroup of K , and for $n \in \mathbb{Z}$, set $\mathfrak{k}_n = \varpi^n \mathfrak{k}$. In particular, there are natural maps

$$K_d/K_{d+1} \simeq \mathfrak{k}_d/\mathfrak{k}_{d+1} \quad \text{and} \quad (\mathfrak{k}_d/\mathfrak{k}_{d+1})^\wedge = \mathfrak{k}_{-d}/\mathfrak{k}_{-d+1} \simeq \mathfrak{gl}(2, \mathbb{F}_q) .$$

The inclusion $\mathfrak{sl}(2, \mathbb{F}_q) \subset \mathfrak{gl}(2, \mathbb{F}_q)$ allows us to canonically extend the previous subsection characters ϕ_Ξ of \mathcal{K}_d (trivial \mathcal{K}_{d+1}), to characters of ZK_d (trivial $Z\mathcal{K}_{d+1}$). We do this, and consequently get a canonical extension of the character τ to ZK_d . We shall use the same symbol τ to denote the extension. We recall:

$$\text{Res}_{\text{SL}(2, F)}^{\text{GL}(2, F)} (\text{c-Ind}_{ZK_d}^{\text{GL}(2, F)} (\tau)) = \text{c-Ind}_{\mathcal{K}_d}^{\text{SL}(2, F)} (\tau) \oplus \text{c-Ind}_{\mathcal{K}'_d}^{\text{SL}(2, F)} (\tau') , \quad (6.2.2)$$

and therefore,

$$e_d^{\text{cusp}} = \frac{1}{\text{meas}(\mathcal{K})} \cdot \text{restriction to } \text{SL}(2, F) \text{ of the character of } \text{c-Ind}_{ZK_d}^{\text{GL}(2, F)} (\tau) . \quad (6.2.3)$$

6.3. The projector e_d^{cusp} on split and ramified elliptic tori.

If $y \in G$ is a regular element which is split or ramified elliptic, and π is an irreducible supercuspidal representation of depth d , the Sally-Shalika [SS] character tables lists the character value $\Theta_\pi(y)$ as zero unless the eigenvalues α, α^{-1} of y have $|\alpha - \alpha^{-1}| < q^{-d}$. Furthermore, when $|\alpha - \alpha^{-1}| < q^{-d}$, the value $\Theta_\pi(y)$ depends only on $|\alpha - \alpha^{-1}|$ and d . By (6.1.1) and (6.1.2), it follows:

$$\begin{aligned} e_d^{\text{cusp}}(y) &= \# \text{ irreducible supercuspidals} \cdot \text{formal degree} \cdot \Theta_\pi(y) \\ &= (q-1)(q+1)q^{(d-1)} \frac{1}{\text{meas}(\mathcal{K})} (q-1)q q^{(d-1)} \Theta_\pi(y). \end{aligned} \quad (6.3.1)$$

Case y split: If y is a (compact) split element with eigenvalues α, α^{-1} , then, by Table 2 (page 1235) of Sally-Shalika [SS], the character value of a depth d irreducible supercuspidal (unramified) representation π is:

$$\Theta_\pi(y) = \begin{cases} 0 & \text{when } \alpha \notin 1 + \wp_F^{d+1} \\ \frac{1}{|\alpha - \alpha^{-1}|} - q^d & \text{for } \alpha \in (1 + \wp_F^{d+1}) ; \end{cases}$$

so, we get

$$e_d^{\text{cusp}}(y) = (q-1)(q+1) \begin{cases} 0 & \text{when } \alpha \notin 1 + \wp_F^{d+1}, \text{ i.e., } |1 - \alpha| > \frac{1}{q^{d+1}} \\ (q-1)q q^{2(d-1)} \left(\frac{1}{|\alpha - \alpha^{-1}|} - q^d \right) & \text{for } |1 - \alpha| \leq \frac{1}{q^{d+1}} ; \end{cases}$$

Case y ramified elliptic: If y is a ramified elliptic element with eigenvalues α, α^{-1} , let E be the quadratic extension of F containing α . Then, the character value of a depth d cuspidal (unramified) representation π is

$$\Theta_\pi(y) = \begin{cases} 0 & \text{when } \alpha \notin (1 + \wp_E^{2d+1}) \\ -q^d & \text{for } \alpha \in (1 + \wp_E^{2d+1}), \text{ i.e., } |1 - \alpha| \leq \frac{1}{q^{d+\frac{1}{2}}}, \end{cases}$$

so, we get

$$e_d^{\text{cusp}}(y) = (q-1)(q+1) \begin{cases} 0 & \text{when } \alpha \notin 1 + \wp_E^{2d+1}, \\ (q-1)q q^{2(d-1)} q^d (-1) & \text{for } |1 - \alpha|_E \leq \frac{1}{q^{d+\frac{1}{2}}} . \end{cases}$$

6.4. The projector e_d^{cusp} on unramified elliptic tori. Suppose $y \in G$ is a regular element whose eigenvalues α, α^{-1} belong to the unramified quadratic extension E/F . We deduce from the formula (6.2.3) that

$$e_d^{\text{cusp}}(y) = 0 \quad \text{unless } \alpha \in 1 + \wp_E^d.$$

Subcase $|\alpha - \alpha^{-1}| = q^{-d}$: The element y has a conjugate in the set $K_d \setminus K_{d+1}$. We may and do assume y is in $K_d \setminus K_{d+1}$.

Lemma 6.4.1. *Let ψ be a non-trivial character of \mathbb{F}_q . Suppose $z \in \mathfrak{sl}(2, \mathbb{F}_q)$ is a elliptic element. Then,*

$$\sum_{\substack{e \in \mathfrak{sl}(2, \mathbb{F}_q) \\ e \text{ elliptic}}} \psi(\text{trace}(ze)) = q. \quad (6.4.2)$$

Proof. The orbit of an elliptic element has a representative of the form $\begin{bmatrix} 0 & 1 \\ u & 0 \end{bmatrix}$, with u a non-square. A parametrization of the elliptic elements is:

$$\text{Ad}\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} 0 & 1 \\ u & 0 \end{bmatrix}, \quad \text{with } u \text{ a non-square, } a \text{ non-zero, and } b \text{ arbitrary.}$$

We take $z = \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix}$, with ϵ a non-square. Then, the sum (6.4.2) becomes

$$\begin{aligned} \sum_{\substack{u \text{ non-square} \\ a \neq 0, b}} \psi(ua^{-1} + \epsilon a - \epsilon b^2 ua^{-1}) &= \sum_a \psi(\epsilon a) \sum_u \psi(ua^{-1}) \sum_b \psi(-\epsilon b^2 ua^{-1}) \\ &= \sum_a \psi(\epsilon a) \sum_{v \text{ non-square}} \psi(va) \sum_c \psi(-ac^2). \end{aligned}$$

Let ψ_a be the character $\psi_a(x) = \psi(ax)$, and let sgn denote the quadratic character on \mathbb{F}_q . Then, $\sum_c \psi(-ac^2a) = G(\psi_{-a}, \text{sgn}) = \text{sgn}(-a) G(\psi, \text{sgn})$, a Gauss sum. So, the sum (6.4.2) equals

$$\begin{aligned}
& \sum_{a \neq 0} \psi(\epsilon a) \left(\sum_{v \text{ non-square}} \psi(va) \right) \text{sgn}(-a) G(\psi, \text{sgn}) \\
&= \sum_{a \neq 0} \psi(\epsilon a) \frac{1}{2} \left(-1 - G(\psi_a, \text{sgn}) \right) \text{sgn}(-a) G(\psi, \text{sgn}) \\
&= -\frac{1}{2} \sum_{a \neq 0} \left(\psi(\epsilon a) \text{sgn}(-a) G(\psi, \text{sgn}) + \psi(\epsilon a) q \right) \\
&= -\frac{1}{2} \left(G(\psi, \text{sgn}) \text{sgn}(-1) \text{sgn}(\epsilon) \sum_{a \neq 0} \psi(a) \text{sgn}(a) + \sum_{a \neq 0} \psi(\epsilon a) q \right) \\
&= -\frac{1}{2} \left(-q + (-q) \right) = q
\end{aligned}$$

Here, we have used elementary properties of Gauss sums [IR]. \square

Corollary 6.4.3. *Let τ be the sum (6.1.3) of the elliptic characters of \mathcal{K}_d (modulo \mathcal{K}_{d+1}). If $y \in \mathcal{K}_d \setminus \mathcal{K}_{d+1}$ is an elliptic element whose eigenvalues α, α^{-1} satisfy $|\alpha - \alpha^{-1}| = q^{-d}$, then*

$$\tau(y) = q.$$

As already mentioned above (6.2.3), τ has a canonical extension to ZK_d . Let $\dot{\tau}$ denote the extension of the function τ to $GL(2, F)$ which is zero outside ZK_d . Suppose $y \in \mathcal{K}_d \setminus \mathcal{K}_{d+1}$ is unramified elliptic. Harish-Chandra's formula [HC] for the right side of (6.2.3)

$$\begin{aligned}
e_d^{\text{cusp}}(y) &= \sum_{gZK_d \in GL(2, F)/ZK_d} \dot{\tau}(gyg^{-1}) \quad \text{for } y \in \mathcal{K}_d \setminus \mathcal{K}_{d+1} \text{ elliptic} \\
&= \sum_{gZK_d \in ZK/ZK_d} \dot{\tau}(gyg^{-1}) = \sum_{gZK_d \in ZK/ZK_d} \dot{\tau}(y) = [ZK : ZK_d] \tau(y) \quad (6.4.4) \\
&= (q+1)(q-1)q(q^3)^{(d-1)}\tau(y) = (q+1)(q-1)q^2 q^{3(d-1)}
\end{aligned}$$

Subcase $|\alpha - \alpha^{-1}| < q^{-d}$: Suppose π is an irreducible supercuspidal representation compact induced from the representation κ of \mathcal{K} . Let κ' be the $GL(2, F)$ conjugation of κ to a representation of \mathcal{K}' , and $\pi' = \text{c-Ind}_{\mathcal{K}'}^G(\kappa')$. By Table 2 (page 1235) of Sally-Shalika [SS],

$$\Theta_\pi(y) + \Theta_{\pi'}(y) = -2q^d;$$

and so

$$\begin{aligned}
e_d^{\text{cusp}}(y) &= \frac{(q-1)}{2} (q+1)q^{(d-1)} \cdot (q-1)q^{(d-1)} \cdot (-2)q^d \\
&= (q-1)(q+1)(q-1)q^{3d-1}(-1) \quad \text{when } |\alpha - \alpha^{-1}| < q^{-d}.
\end{aligned}$$

Table 2 $d \geq 0$ integral	
y has eigenvalues α, α^{-1} : value of $\frac{1}{(q-1)(q+1)} e_d^{\text{cusp}}(y)$	
y split	$\begin{cases} 0 & \text{when } \alpha \notin 1 + \wp_F^{d+1}, \text{ i.e., } 1 - \alpha > \frac{1}{q^{d+1}} \\ (q-1) q q^{2(d-1)} \left(\frac{1}{ \alpha - \alpha^{-1} } - q^d \right) & \text{for } 1 - \alpha \leq \frac{1}{q^{d+1}} \end{cases}$
y ramified elliptic	$\begin{cases} 0 & \text{when } \alpha \notin 1 + \wp_E^{2d+1}, \text{ i.e., } 1 - \alpha > \frac{1}{q^{d+\frac{1}{2}}} \\ (q-1) q q^{2(d-1)} q^d (-1) & \text{for } 1 - \alpha _E \leq \frac{1}{q^{d+\frac{1}{2}}} \end{cases}$
y unramified elliptic	$\begin{cases} 0 & \text{when } \alpha \notin 1 + \wp_E^{d+1}, \text{ i.e., } 1 - \alpha > \frac{1}{q^d} \\ q^{(3d-1)} & \text{when } 1 - \alpha = \frac{1}{q^d} \\ (q-1) q^{(3d-1)} (-1) & \text{for } 1 - \alpha < \frac{1}{q^d} \end{cases}$

7. HALF-INTEGRAL DEPTH SUPERCUSPIDAL REPRESENTATIONS

We abbreviate the Iwahori subgroup $G_{x_{01}}$ and its filtration subgroups $G_{x_{01},r}$ as \mathcal{I} and \mathcal{I}_r respectively.

Suppose $d \in \frac{1}{2} + \mathbb{N}$ is a positive half-integer. A Bernstein component Ω of depth d is the equivalence class of an irreducible supercuspidal representation π . Set

$$d^+ := d + \frac{1}{2}.$$

We recall:

- (i) The group $\mathcal{I}_d/\mathcal{I}_{d^+}$ has $(q-1)^2$ non-degenerate characters. Under the adjoint action of \mathcal{I} , these non-degenerate characters are partitioned into $2(q-1)$ orbits with $\frac{(q-1)}{2}$ characters in an orbit.

- (ii) If (π, V_π) is an irreducible supercuspidal representation, and its depth $\rho(\pi)$ equals d , then the subspace of \mathcal{I}_{d+} -fixed vectors, $V_\pi^{\mathcal{I}_{d+}}$, is non-zero, and is, since \mathcal{I} normalizes the subgroup \mathcal{I}_{d+} , \mathcal{I} -invariant. The characters ϕ_X of \mathcal{I}_d (modulo \mathcal{I}_{d+}) which appear in $V_\pi^{\mathcal{I}_{d+}}$ are non-degenerate. By Clifford theory, the set

$$\{ \phi_\Xi \mid \phi_\Xi \text{ appears in } V_\pi^{\mathcal{I}_{d+}} \}$$

is a single \mathcal{I} -orbit.

The formal degree of π is:

$$d_\pi = \frac{(q+1)}{\mathrm{meas}(\mathcal{K})} \frac{(q-1)}{2} q^{d-\frac{1}{2}}.$$

- (iii) For any non-degenerate character ϕ_Ξ of $\mathcal{I}_d/\mathcal{I}_{d+}$, the compactly supported induced representation

$$\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G(\phi_\Xi)$$

is a finite length (completely reducible) supercuspidal representation. If (π, V_π) is an irreducible supercuspidal representation as in part (i), i.e., $V_\pi^{\mathcal{I}_{d+}}$ contains the (non-degenerate) character ϕ_Ξ , then by Frobenius reciprocity:

$$\mathrm{Hom}_G(V_\pi, \mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G(\phi_\Xi)) \neq \{0\}.$$

Furthermore:

- Up to isomorphism, $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G(\phi_\Xi)$ contains $2q^{(d-\frac{1}{2})}$ distinct classes of irreducible supercuspidal representations (σ, V_σ) .
 - The multiplicity in $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G(\phi_\Xi)$ of any (σ, V_σ) occurring in it is $q^{(d-\frac{1}{2})}$; in particular, the multiplicity is independent of σ .
- (iv) For a non-degenerate character ϕ_Ξ , let S_Ξ denote the set of these $2q^{(d-\frac{1}{2})}$ classes of irreducible supercuspidal representations, and let Θ_Ξ denote the character of the representation $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G(\phi_\Xi)$. We have:

$$\Theta_\Xi = q^{(d-\frac{1}{2})} \sum_{\sigma \in S_\Xi} \Theta_\sigma.$$

By Harish-Chandra's character formula [HC] for induction from an open compact subgroup, Θ_Ξ is supported on $\mathrm{Ad}(G)(\mathcal{I}_d)$. If the cosets Ξ and Ξ' belong to the same \mathcal{I} -orbit, then $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G(\phi_\Xi)$ and $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G(\phi_{\Xi'})$ are equivalent representations, and so $\Theta_\Xi = \Theta_{\Xi'}$. Set

$$\tau := \sum_{\Xi} \phi_\Xi \quad \text{the sum of the non-degenerate characters of } \mathcal{I}_d \text{ (modulo } \mathcal{I}_{d+}), \quad (7.1)$$

and let Θ_τ denote the character of the (compactly supported) induced representation $\mathrm{c}\text{-Ind}_{\mathcal{I}_d}^G(\tau)$. We have:

$$\begin{aligned}
\Theta_\tau &= \sum_{\Xi} \Theta_\Xi = \sum_{\Xi} q^{(d-\frac{1}{2})} \sum_{\sigma \in S_\Xi} \Theta_\sigma \\
&= \frac{(q-1)}{2} q^{(d-\frac{1}{2})} \sum_{\rho(\sigma)=d} \Theta_\sigma .
\end{aligned} \tag{7.2}$$

Whence,

$$\Theta_\tau = \frac{\text{meas}(\mathcal{K})}{(q+1)} \left(\frac{(q+1)}{\text{meas}(\mathcal{K})} \frac{(q-1)}{2} q^{(d-\frac{1}{2})} \right) \sum_{\rho(\sigma)=d} \Theta_\sigma ,$$

i.e.,

$$e_d = \frac{(q+1)}{\text{meas}(\mathcal{K})} \Theta_\tau .$$

As already mentioned above for Θ_Ξ , by Harish-Chandra's formula [HC] for supercuspidal representations obtained via compact induction, we have:

$$\text{support}(e_d) \subset \text{Ad}(G) (\mathcal{I}_d) .$$

This support condition allows us to compute e_d rather efficiently. Note for $d > 0$, and residual characteristic p odd, the set \mathcal{I}_d is contained in \mathcal{U}^{top} . Whence the support of e_d is within the set of topologically unipotent elements.

Suppose $y \in G$ is regular semisimple element. Let α, α^{-1} be the roots of the characteristic polynomial of y . By the support condition:

$$e_d(y) = 0 \quad \text{when} \quad |\alpha - \alpha^{-1}| > q^{-d} .$$

When $|\alpha - \alpha^{-1}| \leq q^{-d}$, we consider three cases for y : split, elliptic unramified and elliptic ramified.

Case y split or elliptic unramified: Here, the eigenvalues of y belong to either F or an unramified quadratic extension, and therefore $|\alpha - \alpha^{-1}|$ is a (positive) integral power of $\frac{1}{q}$, so $|\alpha - \alpha^{-1}| \leq q^{-d}$ in fact means $|\alpha - \alpha^{-1}| < q^{-d}$. By the Sally-Shalika character tables [SS], if π is an irreducible supercuspidal representation of depth d :

$$\Theta_\pi(y) = \begin{cases} 0 & y \text{ split or unramified elliptic, and } |\alpha - \alpha^{-1}| > q^{-d} \\ \frac{1}{|\alpha - \alpha^{-1}|} - \frac{1}{2} q^{d+\frac{1}{2}} \left(\frac{q+1}{q} \right) & \text{when } y \text{ is split and } |\alpha - \alpha^{-1}| \leq q^{-d} \\ -\frac{1}{2} q^{d+\frac{1}{2}} \left(\frac{q+1}{q} \right) & y \text{ unramified elliptic, and } |\alpha - \alpha^{-1}| \leq q^{-d} \end{cases} \tag{7.3}$$

So,

$$\frac{\text{meas}(\mathcal{K})}{(q-1)(q+1)} e_d(y) = \begin{cases} 0 & y \text{ split or unramified elliptic, and } |\alpha - \alpha^{-1}| > q^{-d} \\ \frac{(2(q-1) q^{2(d-\frac{1}{2}})})}{|\alpha - \alpha^{-1}|} - (q-1)(q+1) q^{3(d-\frac{1}{2})} & \text{when } y \text{ is split and } |\alpha - \alpha^{-1}| \leq q^{-d} \\ (q-1)(q+1) q^{3(d-\frac{1}{2})} (-1) & y \text{ unramified elliptic and } |\alpha - \alpha^{-1}| \leq q^{-d} \end{cases}$$

Case y ramified: As already mentioned, by the support condition, we may and do assume $|\alpha - \alpha^{-1}| \leq q^{-d}$. We consider two subcases depending on whether:

$$|\alpha - \alpha^{-1}| < q^{-d} \quad \text{or} \quad |\alpha - \alpha^{-1}| = q^{-d}.$$

Subcase $|\alpha - \alpha^{-1}| = q^{-d}$: The element y has a conjugate in the set $\mathcal{I}_d \setminus \mathcal{I}_{d+\frac{1}{2}}$. We may and do assume y is in $\mathcal{I}_d \setminus \mathcal{I}_{d+\frac{1}{2}}$. We remark that under the isomorphism of $\mathcal{I}_d/\mathcal{I}_{d+\frac{1}{2}}$ with $\mathfrak{g}_{x_{01},d}/\mathfrak{g}_{x_{01},d+\frac{1}{2}}$, the coset y is a non-degenerate coset.

Lemma 7.4. *Let ψ be a non-trivial character of \mathbb{F}_q . Suppose $u, v \in (\mathbb{F}_q)^\times$. Then,*

$$\sum_{a, b \in (\mathbb{F}_q)^\times} \psi(ua) \psi(vb) = 1.$$

Proof. By changes of variables

$$\sum_{a, b \in (\mathbb{F}_q)^\times} \psi(ua) \psi(vb) = \left(\sum_{c \in (\mathbb{F}_q)^\times} \psi(c) \right)^2 = (-1)^2 = 1.$$

□

Corollary 7.5. *If $y \in \mathcal{I}_d \setminus \mathcal{I}_{d+\frac{1}{2}}$ is a ramified elliptic element whose eigenvalues α, α^{-1} satisfy $|\alpha - \alpha^{-1}| = q^{-d}$, then*

$$\tau(y) = 1.$$

Let $\dot{\tau}$ denote the extension of τ to a function on G which is zero outside \mathcal{I}_d . Harish-Chandra's formula [HC] for the character Θ_τ is:

$$\Theta_\tau(x) = \sum_{g\mathcal{I}_d \in G/\mathcal{I}_d} \dot{\tau}(gxg^{-1}).$$

Since $y \in \mathcal{I}_d$ has the property that under the isomorphism $\mathcal{I}_d/\mathcal{I}_{d+\frac{1}{2}} \simeq \mathfrak{g}_{x_{01},d}/\mathfrak{g}_{x_{01},d+\frac{1}{2}}$ it corresponds to a non-degenerate coset, the only $g\mathcal{I}_d$ satisfying $gyg^{-1} \in \mathcal{I}_d$ is when $g \in \mathcal{I}$.

Whence,

$$\begin{aligned}\Theta_\tau(y) &= \sum_{g\mathcal{I}_d \in G/\mathcal{I}_d} \dot{\tau}(gyg^{-1}) = \sum_{g\mathcal{I}_d \in \mathcal{I}/\mathcal{I}_d} \dot{\tau}(gyg^{-1}) \\ &= [\mathcal{I} : \mathcal{I}_d] \dot{\tau}(y) = (q-1)q^{3(d-\frac{1}{2})}\tau(y) = (q-1)q^{3(d-\frac{1}{2})};\end{aligned}$$

and

$$\frac{\text{meas}(\mathcal{K})}{(q-1)(q+1)} e_d(y) = q^{3(d-\frac{1}{2})} \quad \text{when} \quad |\alpha - \alpha^{-1}| = q^{-d}.$$

Subcase $|\alpha - \alpha^{-1}| < q^{-d}$: We note irreducible supercuspidal representations of depth d come in pairs π and π' , i.e., an L -packet. In the Sally-Shalika [SS] parameterization of ramified irreducible supercuspidal representation, each element of the pair corresponds to taking one of two classes of additive characters of F , and their character table gives:

$$(\Theta_\pi + \Theta_{\pi'})(y) = -q^{d+\frac{1}{2}} \frac{(q+1)}{q} = -q^{d-\frac{1}{2}}(q+1) \quad \text{when} \quad |1 - \alpha| < q^{-d};$$

so,

$$(e_\pi + e_{\pi'})(y) = \frac{(q-1)(q+1)q^{(d-\frac{1}{2})}}{2 \text{meas}(\mathcal{K})} \cdot (-q^{d-\frac{1}{2}}(q+1)).$$

Whence,

$$\begin{aligned}e_d(y) &= 2(q-1)q^{(d-\frac{1}{2})} \left(\frac{(q-1)(q+1)q^{(d-\frac{1}{2})}}{2 \text{meas}(\mathcal{K})} \right) \cdot (-q^{d-\frac{1}{2}}(q+1)) \\ &= -\frac{(q-1)^2(q+1)^2}{\text{meas}(\mathcal{K})} q^{3(d-\frac{1}{2})},\end{aligned}$$

i.e.,

$$\frac{\text{meas}(\mathcal{K})}{(q-1)(q+1)} e_d(y) = (q-1)(q+1) q^{3(d-\frac{1}{2})} (-1).$$

Table 3 $d \in \mathbb{N}$ half-integral	
y has eigenvalues α, α^{-1} : value of $\frac{1}{(q-1)(q+1)} e_d(y)$	
y split	$\begin{cases} 0 & \text{when } \alpha \notin 1 + \wp_F^{d+\frac{1}{2}}, \text{ i.e., } 1 - \alpha > \frac{1}{q^d} \\ \frac{(2(q-1) q^{2(d-\frac{1}{2}})})}{ \alpha - \alpha^{-1} } - (q-1)(q+1) q^{3(d-\frac{1}{2})} & \text{when } \alpha - \alpha^{-1} \leq q^{-d} \end{cases}$
y ramified elliptic	$\begin{cases} 0 & \text{when } 1 - \alpha > \frac{1}{q^d} \\ q^{3(d-\frac{1}{2})} & \text{for } 1 - \alpha _E = \frac{1}{q^d} \\ (q-1)(q+1) q^{3(d-\frac{1}{2})} (-1) & \text{for } 1 - \alpha _E < \frac{1}{q^d} \end{cases}$
y unramified elliptic	$\begin{cases} 0 & \text{when } 1 - \alpha > \frac{1}{q^d} \\ (q-1)(q+1) q^{3(d-\frac{1}{2})} (-1) & \text{for } 1 - \alpha < \frac{1}{q^d} \end{cases}$

8. THE MAIN RESULT

For convenience in numbering, for $k \in \frac{1}{2}\mathbb{N}$, set

$$\sigma_k := e_0 + e_{\frac{1}{2}} + \cdots + e_k. \quad (8.1)$$

We note in particular $\sigma_0 = e_0$.

Theorem 8.2. *For $k \in \frac{1}{2}\mathbb{N}$, set $k^+ := k + \frac{1}{2}$. Under the assumption the p -adic field F has odd residue characteristic, we have $\text{supp}(\sigma_k) \subset \mathcal{U}_{k^+}^{\text{top}}$. On $\mathcal{U}_{k^+}^{\text{top}}$:*

- *When k is integral:*

$$\sigma_k(y) = (q^2 - 1) q^{3k} \begin{cases} \left(\frac{2q^{-k}}{|\alpha - \alpha^{-1}|_F} - 1 \right) & \text{when } y \text{ is split with eigenvalues } \alpha, \alpha^{-1} \\ -1 & \text{when } y \text{ is elliptic} \end{cases}$$

- When k is half-integral:

$$\sigma_k(y) = (q^2 - 1) q^{3k + \frac{1}{2}} \begin{cases} \left(\frac{2q^{-(k + \frac{1}{2})}}{|\alpha - \alpha^{-1}|_F} - 1 \right) & \text{when } y \text{ is split with eigenvalues } \alpha, \alpha^{-1} \\ -1 & \text{when } y \text{ is elliptic} \end{cases}$$

Proof. The proof is induction on the depth: The values of $\sigma_0 = e_0$ are given in Table 1. Given σ_k , we compute $\sigma_{k+} = \sigma_k + e_{k+}$ via Table 3 when k is integral, and via (4.3) and Table 2 when k is half-integral. \square

For $k \in \frac{1}{2}\mathbb{N}$, we note the series defining the exponential and logarithm maps between \mathfrak{g} and G converge for k sufficiently large. We take $k_0 \in \frac{1}{2}\mathbb{N}$ so that:

$$\exp \text{ and } \log \text{ are bijections between } \mathfrak{g}_k \text{ and } G_k \text{ when } k \geq k_0. \quad (8.3)$$

Corollary 8.4. *Under condition (8.3), so that $\sigma_k \circ \exp$ is defined and has support in $\mathcal{N}_{k+}^{\text{top}}$, if $Y \in \mathcal{N}_{k+}^{\text{top}}$ has eigenvalues $\pm \lambda$, then :*

- When k is integral and $Y \in \mathfrak{g}_{k+}$:

$$\sigma_k \circ \exp(Y) = (q^2 - 1) q^{3k} \begin{cases} \left(\frac{2q^{-k}}{|\lambda|_F} - 1 \right) & \text{when } y \text{ is split} \\ -1 & \text{when } y \text{ is elliptic} \end{cases}$$

- When k is half-integral and $Y \in \mathfrak{g}_{k+}$:

$$\sigma_k \circ \exp(Y) = (q^2 - 1) q^{3k + \frac{1}{2}} \begin{cases} \left(\frac{2q^{-(k + \frac{1}{2})}}{|\lambda|_F} - 1 \right) & \text{when } y \text{ is split} \\ -1 & \text{when } y \text{ is elliptic} \end{cases}$$

In particular, $\sigma_{k+1} \circ \exp$ and $\sigma_k \circ \exp$ satisfy the homogeneity relation:

$$(\sigma_{k+1} \circ \exp)(\varpi Y) = q^3 (\sigma_k \circ \exp)(Y). \quad (8.5)$$

Remarks: (i) Examination of the explicit formulae for σ_k and $\sigma_k \circ \exp$, shows these distributions depend only on the characteristic polynomial of the input, and therefore are stable distributions.

(ii) The formula for $\sigma_0 = \epsilon_0$ leads to the observation that it is the restriction of the Steinberg character to $\mathcal{U}^{\mathrm{top}}$.

(iii) The transfer of σ_k to $\sigma_k \circ \exp$ is only valid when $k \geq k_0$, but the homogeneity relation (8.5) allows us to formally continue $\sigma_k \circ \exp$ to the range $0 \leq k < k_0$, so the continuation for parameter has support in \mathfrak{g}_{k+} , e.g., $\mathfrak{g}_{0+} = \mathcal{N}^{\mathrm{top}}$ when $k = 0$. This is analogous to the behavior of σ_k 's in the same range.

(iv) The power 3 of the factor q^3 should be viewed as the dimension of \mathfrak{g} . Under Fourier transform on the Lie algebra, the homogeneity relation (8.5) becomes a

$$\mathrm{FT}(\sigma_{k+1} \circ \exp)(\varpi^{-1}Y) = \mathrm{FT}(\sigma_k \circ \exp)(Y). \quad (8.6)$$

To identify $\mathrm{FT}(\sigma_k \circ \exp)$, we note the following Proposition, whose proof is in the appendix:

Proposition. A.1 *For $\mathfrak{g} = \mathfrak{sl}(2, F)$, we have*

- The Fourier transforms $\mathrm{FT}(1_{\mathfrak{g}_0})$ and $\mathrm{FT}(1_{\mathfrak{g}_{-\frac{1}{2}}})$ have support in the sets $\mathfrak{g}_{0+} := \mathfrak{g}_{\frac{1}{2}}$ and $\mathfrak{g}_{(\frac{1}{2})+} := \mathfrak{g}_1$ respectively. In particular, the support is contained in $\mathcal{N}^{\mathrm{top}}$.
- For $k \geq 1$, the Fourier transform $\mathrm{FT}(1_{\mathfrak{g}_{-k}})$ has support in $\mathfrak{g}_{k+} := \mathfrak{g}_{k+\frac{1}{2}}$.

For a general connected reductive p-adic group, under conditions in which the exponential map takes $\mathcal{N}_r^{\mathrm{top}}$ to $\mathcal{U}_r^{\mathrm{top}}$ ($r > 0$), Kim [Ka, Kb], showed, for X in $\mathfrak{g}_{(\frac{d}{2})+}$:

$$\int_{\widehat{G}_{\leq d}^{\mathrm{temp}}} \Theta_{\pi}(\exp(X)) d\mu_{\mathrm{PM}}(\pi) = \mathrm{FT}(1_{\mathfrak{g}_{-d}})(X),$$

where $\widehat{G}_{\leq d}^{\mathrm{temp}}$ is the (classes of) irreducible tempered representations of depth less than or equal to d . In this situation, for $\mathrm{SL}(2)$, we have

$$\sigma_d \circ \exp = \mathrm{FT}(1_{\mathfrak{g}_{-d}}) \quad (\text{both sides have support in } \mathfrak{g}_{d+}).$$

We conjecture, for $\mathrm{SL}(2)$, and more generally for any connected reductive p-adic group, this identity is true when the depth is sufficiently large.

APPENDIX A.

Here $G = \mathrm{SL}(2, F)$, and $\mathfrak{g} = \mathfrak{sl}(2, F)$. Suppose ψ is an additive character of F with conductor \wp . Let FT denote the Fourier transform on \mathfrak{g} , i.e., if $f \in C_c^\infty(\mathfrak{g})$:

$$\mathrm{FT}(f)(Y) = \int_{\mathfrak{g}} \psi(\mathrm{trace}(XY)) f(X) dX.$$

In this appendix we prove:

Proposition A.1. *For $\mathfrak{g} = \mathfrak{sl}(2, F)$, we have*

- *The Fourier transforms $\mathrm{FT}(1_{\mathfrak{g}_0})$ and $\mathrm{FT}(1_{\mathfrak{g}_{-\frac{1}{2}}})$ have support in the sets $\mathfrak{g}_{0+} := \mathfrak{g}_{\frac{1}{2}}$ and $\mathfrak{g}_{(\frac{1}{2})+} := \mathfrak{g}_1$ respectively. In particular, the support is contained in $\mathcal{N}^{\mathrm{top}}$.*
- *For $k \geq 1$, the Fourier transform $\mathrm{FT}(1_{\mathfrak{g}_{-k}})$ has support in $\mathfrak{g}_{k+} := \mathfrak{g}_{k+\frac{1}{2}}$.*

Proof. Since $1_{\mathfrak{g}_0}$, and $1_{\mathfrak{g}_{-\frac{1}{2}}}$ are $\mathrm{Ad}(G)$ -invariant sets, their Fourier transforms are $\mathrm{Ad}(G)$ -invariant. Therefore, it is sufficient to show the stated vanishing on any convenient element in an Adjoint orbit.

We prove the result for $1_{\mathfrak{g}_0}$ and remark our argument proof is easily adapted to also treat the case $1_{\mathfrak{g}_{-\frac{1}{2}}}$. We have

$$\mathrm{FT}(1_{\mathfrak{g}_0})(Y) = \mathrm{PV} \int_{\mathfrak{g}} \psi(\mathrm{trace}(XY)) 1_{\mathfrak{g}_0}(X) dX \quad (\text{principal value}).$$

We note that $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid \det(X) \in \mathcal{R}_F\}$. Let $X = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, and for integral ℓ , set

$$\mathcal{T}_\ell = \{X \in \mathfrak{g}_0 \mid a, b, c \in \wp^{-\ell}\} = \{X \in \mathfrak{g} \mid \det(X) \in \mathcal{R}_F \text{ and } a, b, c \in \wp^\ell\}.$$

We show for $Y \notin \mathfrak{g}_{\frac{1}{2}}$ the integral

$$\int_{\mathcal{T}_\ell} \psi(\mathrm{trace}(XY)) 1_{\mathfrak{g}_0}(X) dX = \int_{\mathcal{T}_\ell} \psi(\mathrm{trace}(XY)) dX \quad \text{vanishes for } \ell \text{ large.}$$

The Fourier transforms $\mathrm{FT}(1_{\mathfrak{g}_j})$ are invariant under the Adjoint action of $\mathrm{GL}(2, F)$. The $\mathrm{GL}(2, F)$ -orbit of a regular semisimple element Y contains an element of the form

$$\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \quad \text{with } \mathrm{val}(B) \leq \mathrm{val}(C) \leq \mathrm{val}(B) + 1.$$

We take Y to have this anti-diagonal form. Here, Y is not topologically nilpotent when $\mathrm{val}(C) \leq 0$. It thus suffices to prove $\mathrm{FT}(1_{\mathfrak{g}_0})(Y) = 0$ in this situation. For $X = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, we have $\mathrm{trace}(XY) = (bC + cB)$, and so

$$\int_{\mathcal{T}_\ell} \psi(\mathrm{trace}(XY)) dX = \int_{\mathcal{T}_\ell} \psi(bC + cB) dX.$$

CASE $\text{val}(C) = \text{val}(B)$: We remark in this situation, Y is either split or elliptic unramified. We show the integral vanishes if $B, C \notin \wp$. Our strategy is to partition \mathcal{T}_ℓ into regions where the integral is zero.

SUBCASE $a \in \mathcal{R}_F$: The condition for X to be in \mathfrak{g}_0 is $a^2 + bc \in \mathcal{R}_F$, and so $bc \in \mathcal{R}_F$. We consider two subcases based on b .

- Subcase $b \in \mathcal{R}$. Here, dependent on the valuation $\text{val}(b)$, the variable c runs over an ideal between \wp^0 and $\wp^{-\ell}$. The assumption $B \notin \wp$ means $c \rightarrow \psi(cB)$ is a non-trivial character on its allowed ideal and therefore, for fixed a and b , the integral over c is zero. We deduce the integral over the region in \mathcal{T}_ℓ which satisfies $a, b \in \mathcal{R}$ is zero.
- Subcase $b = \varpi^{-k}u$ with (integral) $k > 0$ and u a unit. The condition $bc \in \mathcal{R}$, is $c \in \wp^k \subset \wp$. If $\text{val}(B) \leq -k$, then $c \rightarrow \psi(cB)$ is a non-trivial character; so integration of c over \wp^k is zero. If $-k < \text{val}(B)$, then $\psi(cB) = 1$; so $\psi(bc + cB) = \psi(bC)$. Integration over $c \in \wp^k$ yields $\psi(bC) \text{meas}(\wp^k)$. We note that $x \rightarrow \psi(xC)$ is a non-trivial character on \mathcal{R} , and $\psi((b+x)C) = \psi(bC)\psi(xC)$. If we integrate over all $b \in \wp^{-k} \setminus \wp^{-k+1}$ we get zero. We deduce the integral over the region in \mathcal{T}_ℓ which satisfies $a \in \mathcal{R}$ and $b \notin \mathcal{R}$ is zero.

CASE $a \notin \mathcal{R}_F$: Write a as $a = \varpi^{-k}u$ with u a unit, and k a positive integer (note $k \leq \ell$ to satisfy $a \in \wp^{-\ell}$). The condition $a^2 + bc \in \mathcal{R}$ for X to be in \mathfrak{g}_0 is thus $-bc \in \varpi^{-2k}u^2 + \mathcal{R}$. In particular, b and c are non-zero. Write b , as $b = \varpi^\beta v_b$, with v_b a unit. The condition, $b, c \in \wp^{-\ell}$ means $-\ell \leq \beta \leq \ell - 2k$, and similarly for the valuation $\gamma = \text{val}(c)$ of c .

- If $\beta \neq \gamma$, then by the symmetry of b and c , we assume $\beta < -k < \gamma$. This imposes the condition $c \in -\varpi^\gamma v_b^{-1}u^2 + \wp^{-\beta}$.
- If $\text{val}(B) - \beta \leq 0$, then for fixed a and b , the integral over c is zero.
- If $\text{val}(B) - \beta > 0$, then $\psi(cB) = \psi(-\varpi^\gamma v_b^{-1}u^2 B)$ is independent of $c \in -\varpi^\gamma v_b^{-1}u^2 + \wp^{-\beta}$. Thus, if we fix a and b , and integrate $\psi(bc + cB)$ over c , we get

$$\psi(bC) \psi(-\varpi^\gamma v_b^{-1}u^2 B) \text{meas}(\varpi^{(-2k-\beta)} + \wp^{-\beta}).$$

If we perturb b by $v_b x \in \wp^{-\text{val}(B)}$ to $b' = b + v_b x$, so $v_{b'} = v_b(1 + \varpi^{-\beta}x)$, then the corresponding c' satisfies $c' \in -\varpi^\gamma v_{b'}^{-1}(1 + \varpi^{-\beta}x)^{-1}u^2 + \wp^{-\beta} = -\varpi^\gamma v_b^{-1}u^2 + \varpi^{\gamma-\beta}v_b^{-1}xu^2 + \wp^{-\beta}$. We deduce $\psi(c'B) = \psi(-\varpi^\gamma v_b^{-1}u^2 B)$ is independent of x . So, $\psi(b'C + c'B) = \psi(bC) \psi(xC) \psi(-\varpi^\gamma v_b^{-1}u^2 B)$, and therefore, if we restrict to $b \in \varpi^{-\beta}\mathcal{R}^\times$ integrate over c' followed by integration over b , we get zero.

- If $\beta = \gamma = -k$, we have

$$X = \varpi^{-k} \begin{bmatrix} u & v_b \\ v_c & -u \end{bmatrix} \quad \text{with } u, v_b, v_c \text{ units,}$$

and the condition for $X \in \mathfrak{g}_0$ is $u^2 + v_b v_c \in \wp^{2k}$. We see the product $-v_b v_c$ must be a square in \mathcal{R}^\times . Conversely, if $-v_b v_c$ is a square, the condition on u is $u^2 \in -v_b v_c + \wp^{2k}$. We fix v_c and multiplicatively perturb v_b by $1 + \wp$, the integral of $\psi(\varpi^{-k}(v_c B + v_{b'} C))$ over $b' \in \varpi^{-k}v_b(1 + \wp)$, and $a = \varpi^{-k}u$ with $u^2 \in -v_{b'} v_c + \wp^{2k}$. The integration over a yields a constant independent of b' , and then the integration over b' is zero. So the

integral of $\psi(\varpi^{-k}(v_c B + v_b C))$ over the region satisfying $X \in \mathfrak{g}_0$ and $a, b, c \in \varpi^{-k}\mathcal{R}^\times$ is zero.

CASE $\text{val}(C) = \text{val}(B) + 1$: The proof here is a minor modification of the case $\text{val}(C) = \text{val}(B)$. We omit the details.

This completes the proof that $\text{FT}(1_{\mathfrak{g}_0})$ has support in $\mathfrak{g}_{\frac{1}{2}}$.

The statement about the support of $\text{FT}(1_{\mathfrak{g}_{-k}})$ for $k \geq 0$ follows from the elementary property $\mathfrak{g}_{k-1} = \varpi^{-1}\mathfrak{g}_k$, and elementary homogeneities of the Fourier transform. \square

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